

Adaptive sampled-data based linear quadratic optimal control of stochastic systems

S. TAN[†] and J. F. ZHANG^{*‡}

[†]National Laboratory of Space Intelligent Control, Beijing Institute of Control Engineering,
P.O. Box 2729, Beijing 100080, China

[‡]Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing 100080, China

(Received 19 July 2006; in final form 1 June 2007)

The problem of sampled-data (SD) based adaptive linear quadratic (LQ) optimal control is considered for linear stochastic continuous-time systems with unknown parameters and disturbances. To overcome the difficulties caused by the unknown parameters and incompleteness of the state information, and to probe into the influence of sample size on system performance, a cost-biased parameter estimator and an adaptive control design method are presented. Under the assumption that the unknown parameter belongs to a known finite set, some sufficient conditions ensuring the convergence of the parameter estimate are obtained. It is shown that when the sample step size is small, the SD-based adaptive control is LQ optimal for the corresponding discretized system, and sub-optimal compared with that of the case where the parameter is known and the information is complete.

1. Introduction

Practically, many control systems that are implemented today are based on sampled-data (SD) control (Åström and Wittenmark 2002), so the SD-based control problem has received considerable attention. For such systems, the available information for control design is the measurement of the system state at the sample time instance rather than the complete state process. Some fundamental and elegant results on SD-based control systems have been obtained on controllability and observability (Fúster 1991), stabilization (Nešić *et al.* 1999, Ishii and Francis 2003), the H_∞ problem (Toivonen and Sãgfors 1997, Bamieh and Pearson 1992), robust and adaptive control (Zhang *et al.* 1989, Ortega and Kreisselmeier 1990, Hu and Holot 1993, Ilchmann and Townley 1999) and optimal control (Qiu and Chen 1994, Yao and Zhang 2003). Among others, the most problems of concern are related to the adaptive control and optimal control. In the adaptive

control case, the systems are with unknown parameters or uncertainties, and the works are mainly devoted to SD-based control design and stabilization analysis of the closed-loop systems (Zhang *et al.* 1989, Ortega and Kreisselmeier 1990, Hu and Holot 1993, Ilchmann and Townley 1999). As for the SD-based optimal control, the works mainly focus on the case where the parameters are known and the system has no disturbance (Qiu and Chen 1994, Yao and Zhang 2003, Tan *et al.* 2005). For instance, Yao and Zhang (2003) investigated the optimality of the SD-based LQ control for linear stochastic continuous-time systems with known parameters; Tan *et al.* (2005) presented an SD-based adaptive LQ control design procedure for systems with unknown Markov jump parameters and without disturbance, and studied the stability and index optimality of the closed-loop systems. It is shown that when the sample step size is small, the SD-based adaptive LQ control is suboptimal. Although some work has been done with the SD-based system, to the authors' knowledge, there is no SD-based adaptive LQ optimal control result about

*Corresponding author. Email: jif@iss.ac.cn

the invariant linear systems with unknown parameters and Brownian motion noise yet.

In this paper, we would like to study the SD-based adaptive LQ optimal control problem for linear stochastic continuous-time systems with both unknown parameters and disturbances. Similar to Kumar (1983), we assume that the unknown parameter belongs to a known finite set. In this SD-based control case, it is very difficult and complicated to construct an adaptive LQ optimal control due to the existence of the unknown parameters and disturbances, and incomplete state information, since the conventional full-data based optimal (adaptive) control (see Caines and Zhang 1995, Duncan *et al.* 1999, Chen *et al.* 1996) is not feasible. Thus, some problems emerge naturally, such as, whether or not there exists an SD-based adaptive LQ optimal control when the system is with unknown parameters and disturbances, how to construct an SD-based adaptive optimal control, and what is the cost difference between the full-data-based optimal control and the SD-based optimal control. To answer these questions, we will first estimate the unknown parameters by using a cost-biased least square algorithm inspired by Kumar (1983), and then, design an SD-based adaptive control according to the certainty equivalent principle. It is shown that when the sample step size is small, the SD-based adaptive control is LQ optimal for the corresponding discretized system, and sub-optimal compared with that of the case where the parameter is known and the complete information rather than only the sampled-data of the state process of the continuous-time system is available for control design.

The remainder of this paper is organized as follows: in §2, the control system to be studied is described, and some assumptions, notations and preliminary results are given. In §3, a parameter estimation method is presented, and an SD-based adaptive LQ control is designed. Besides, the convergence property of the parameter estimate is analysed. Section 4 is concerned with the stability analysis of the closed-loop system with the SD-based adaptive control. Section 5 studies the optimality of the SD-based adaptive control. Section 6 gives some concluding remarks.

2. Problem formulation and preliminary results

Consider a stochastic linear continuous-time system of the form

$$dx_t = A(\theta_0)x_t dt + B(\theta_0)u_t dt + C(\theta_0)dW_t, \quad (1)$$

where $\theta_0 \in \Theta$ is the unknown system parameter to be estimated for control design, Θ is a known finite set, $x_t \in \mathbb{R}^n$ is the state vector, $u_t \in \mathbb{R}^m$ is the control vector and $W_t \in \mathbb{R}^l$ is a standard Brownian motion with

$$EW_t = 0, \quad EW_t W_t^T = tI. \quad (2)$$

W^T denotes the transpose of W , $E(\cdot)$ denotes the mathematical expectation. The initial value x_0 is with normal distribution and $E\|x_0\|^2 < \infty$. $\|\cdot\|$ is the Euclid norm of \mathbb{R}^n and the corresponding induced matrix norm. I is an identity matrix with appropriate dimension.

For expression simplicity, in the sequel, we will denote $A(\theta_0)$, $B(\theta_0)$ and $C(\theta_0)$ by A_0 , B_0 and C_0 , respectively.

The problem we would like to solve is to design a sampled-data (SD) based adaptive control for the system (1) to minimize the following quadratic cost function

$$J(u) = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (x_s^T Q x_s + u_s^T R u_s) ds, \quad (3)$$

where $R > 0$, $Q \geq 0$, and $u \triangleq \{u_t, t \geq 0\}$ with $u_t \in \sigma\{x_s, s \leq t\}$ and $\sigma\{x_s, s \leq t\}$ being the σ -algebra generated by $\{x_s, s \leq t\}$.

Suppose that the sample step is h , and let

$$t' = \left\lfloor \frac{t}{h} \right\rfloor h, \quad (4)$$

where $\lfloor x \rfloor$ denotes the maximal integer less than or equal to x . Then for a continuous-time system of the form

$$dx_t = A(\theta)x_t dt + B(\theta)u_t dt + C(\theta)dW_t \quad (5)$$

with SD-based control

$$u_t = u_{t'}, \quad t \in [t', t' + h), \quad (6)$$

we have

$$x_{(k+1)h} = \bar{A}(\theta)x_{kh} + \bar{B}(\theta)u_{kh} + \bar{W}_{(k+1)h}(\theta), \quad (7)$$

where $x_{kh} = x(kh)$, and

$$\bar{A}(\theta) = e^{A(\theta)h}, \quad \bar{B}(\theta) = \int_0^h e^{A(\theta)s} ds B(\theta), \quad (8)$$

$$\bar{W}_{(k+1)h}(\theta) = \int_{kh}^{(k+1)h} e^{A(\theta)((k+1)h-s)} C(\theta) dW_s. \quad (9)$$

For the convenience of citation, we introduce the following assumptions.

Assumption A1: (A_0, C_0) is controllable.

Assumption A2: For all $\theta \in \Theta$, $(A(\theta), B(\theta), Q^{1/2})$ is controllable and observable.

Assumption A3: The sample step $h < \pi/|\text{Im}\lambda|$, $\forall \lambda \in \Lambda(A(\theta))$ and $\theta \in \Theta$ where $\text{Im}\lambda$ denotes the imaginary part of λ , and $\Lambda(A(\theta))$ denotes the eigenvalue set of $A(\theta)$. When $\text{Im}\lambda = 0$, let $\pi/|\text{Im}\lambda| = \infty$.

Assumption A1 is to guarantee some excitation degree of the disturbance to system state process in order to ensure the convergence of parameter estimate (Chen *et al.* 1996). Many different assumptions on the noise gain matrix can be found in the literature. For instance, it is assumed to be of full rank in Kumar (1983) and Caines and Chen (1985), or $\text{span}(B_0) \subset \text{span}(C_0)$ in Chen (1995) and Gao and Pasik-Duncan (1997).

Assumption A2 is standard for the LQ control problem. It ensures that the following continuous Riccati equation has a unique positive definite solution $P(\theta)$ for every $\theta \in \Theta$:

$$A^T(\theta)P(\theta) + P(\theta)A(\theta) - P(\theta)B(\theta)R^{-1}B^T(\theta)P(\theta) + Q = 0. \quad (10)$$

Assumption A3 together with Assumption A2 guarantees the controllability of $(\bar{A}(\theta), \bar{B}(\theta))$ and the observability of $(\bar{A}(\theta), Q^{1/2})$, and hence, ensures that the following discrete Riccati equation has a unique positive definite solution $\bar{P}(\theta)$ for every $\theta \in \Theta$ (Sontag 1998):

$$\begin{aligned} \bar{P}(\theta) = & \bar{A}^T(\theta) \left[\bar{P}(\theta) - \bar{P}(\theta)\bar{B}(\theta)(\bar{B}^T(\theta)\bar{P}(\theta)\bar{B}(\theta) + R)^{-1} \right. \\ & \left. \times \bar{B}^T(\theta)\bar{P}(\theta) \right] \bar{A}(\theta) + Q. \end{aligned} \quad (11)$$

Let

$$\begin{aligned} K(\theta) &= -R^{-1}B^T(\theta)P(\theta), \\ \bar{K}(\theta) &= -[\bar{B}^T(\theta)\bar{P}(\theta)\bar{B}(\theta) + R]^{-1}\bar{B}^T(\theta)\bar{P}(\theta)\bar{A}(\theta). \end{aligned} \quad (12)$$

Then for any given θ , under Assumptions A2 and A3, from Bertsekas (1976), Kumar (1983) and Sontag (1998) we have the following.

(i) (11) can be rewritten as

$$\begin{aligned} \bar{P}(\theta) = & [\bar{A}(\theta) + \bar{B}(\theta)\bar{K}(\theta)]^T \bar{P}(\theta) [\bar{A}(\theta) + \bar{B}(\theta)\bar{K}(\theta)] \\ & + \bar{K}^T(\theta)R\bar{K}(\theta) + Q \end{aligned} \quad (13)$$

(ii) $\bar{A}(\theta) + \bar{B}(\theta)\bar{K}(\theta)$ is stable in the sense that all eigen-

values of $\bar{A}(\theta) + \bar{B}(\theta)\bar{K}(\theta)$ are in the open unit disk of the complex plane.

(iii) Within the class of matrices K such that $\bar{A}(\theta) + \bar{B}(\theta)K$ is stable, $\bar{K}(\theta)$ is the unique feedback gain to minimize the cost function

$$\bar{J}(\theta, u) = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} (x_{ih}^T(\theta)Qx_{ih}(\theta) + u_{ih}^T(\theta)Ru_{ih}(\theta)), \quad (14)$$

where $u = \{u_{ih}, i = 0, 1, \dots, \}$ with $u_{ih} \in \sigma\{x_{jh}, j = 0, 1, \dots, i\}$ and $\sigma\{x_{jh}, j = 0, 1, \dots, i\}$ being the σ -algebra generated by $\{x_{jh}, j = 0, 1, \dots, i\}$.

(iv) Let

$$\bar{J}(\theta) = \min_u \bar{J}(\theta, u), \quad (15)$$

then

$$\bar{J}(\theta) = \text{tr} \left(\int_0^h C^T(\theta)e^{A^T(\theta)s} \bar{P}(\theta)e^{A(\theta)s} C(\theta) ds \right), \quad (16)$$

where we have used

$$\begin{aligned} E \left[\bar{W}_{(k+1)h}^T(\theta) \bar{P}(\theta) \bar{W}_{(k+1)h}(\theta) \right] \\ = \text{tr} \left(\int_0^h C^T(\theta)e^{A^T(\theta)s} \bar{P}(\theta)e^{A(\theta)s} \times C(\theta) ds \right). \end{aligned} \quad (17)$$

For system (1), it is well-known that when the parameter θ_0 , i.e., (A_0, B_0) , is known *a priori*, and $(A_0, B_0, Q^{1/2})$ is controllable and observable, the algebraic Riccati equation

$$A_0^T P_0 + P_0 A_0 - P_0 B_0 R^{-1} B_0^T P_0 + Q = 0 \quad (18)$$

has a unique positive definite solution P_0 , and the linear feedback

$$u_t^* = -R^{-1}B_0^T P_0 x_t \quad (19)$$

is such that the quadratic cost function (3) reaches its minimal, i.e.,

$$J(u^*) = \min_u J(u) = \text{tr}(C_0^T P_0 C_0). \quad (20)$$

3. Parameter estimation and SD-based adaptive control design

To design an adaptive control law for system (1), it is natural to choose the well-known least square method to estimate the unknown parameters. However, if the

least square method is used alone, then it is sometimes difficult to get an optimal adaptive control.

To see this, let us consider a one-dimension system (1). Suppose its parameter (A, B, C) is unknown, but is known to be in the set $\{(-2 \ln 2, -4 \ln 2, 1), (0, 2, 1)\}$. The cost function is of the form (3) with $Q=1$ and $R=12$.

Here we choose sample step size $h=0.5$. Then, by (7)–(9) we have

$$\begin{aligned} \bar{A}(1) &= 1/2, & \bar{B}(1) &= -1, & \bar{A}(2) &= 1, & \bar{B}(2) &= 1, \\ \bar{W}_{0.5(k+1)}(1) &= \int_{0.5k}^{0.5(k+1)} 2^{-(k+1)+2s} dW_s, \\ \bar{W}_{0.5(k+1)}(2) &= \int_{0.5k}^{0.5(k+1)} dW_s. \end{aligned}$$

From Kumar (1983), the optimal control law of the corresponding discrete-time system is

$$u_{kh} = \bar{K}(\theta_0)x_{kh} = \begin{cases} \frac{2\sqrt{7}-5}{6}x_{kh}, & \text{if } \theta_0 = 1, \text{ i.e. } (\bar{A}, \bar{B}) \\ & = (\frac{1}{2}, -1); \\ -\frac{1}{4}x_{kh}, & \text{if } \theta_0 = 2, \text{ i.e. } (\bar{A}, \bar{B}) \\ & = (1, 1), \end{cases} \quad (21)$$

where $t \in ([kh, (k+1)h])$ and $\bar{K}(\theta_0)$ is given by (12).

Since we do not know the true value of (A, B, C) , at each sample time instant $t=kh$ we will use the least square method to get the estimate $\hat{\theta}_{kh}$ of the true value θ_0 based on the sampled-data $(x_0, u_0, x_h, u_h, \dots, x_{kh})$, and then, use the estimate to design an adaptive control law u_{kh} . To this end, let

$$\begin{aligned} e_1(k) &= \sum_{i=0}^{k-1} (x_{(i+1)h} - \bar{A}(1)x_{ih} - \bar{B}(1)u_{ih})^2 \\ &= \sum_{i=0}^{k-1} (x_{(i+1)h} - \frac{1}{2}x_{ih} + u_{ih})^2, \\ e_2(k) &= \sum_{i=0}^{k-1} (x_{(i+1)h} - \bar{A}(2)x_{ih} - \bar{B}(2)u_{ih})^2 \\ &= \sum_{i=0}^{k-1} (x_{(i+1)h} - x_{ih} - u_{ih})^2. \end{aligned}$$

If $e_1(k) \leq e_2(k)$, then by the least square method, $(\bar{A}(1), \bar{B}(1)) = (\frac{1}{2}, -1)$ is chosen to be the estimate of the true parameter (A, B) , or equivalently, $\hat{\theta}_{kh} = 1$. Otherwise, $(\bar{A}(2), \bar{B}(2)) = (1, 1)$ is chosen to be the

estimate, or $\hat{\theta}_{kh} = 2$. Therefore, by (21) the optimal adaptive control is

$$u_{kh} = \begin{cases} \frac{2\sqrt{7}-5}{6}x_{kh}, & \text{if } e_1(k) \leq e_2(k); \\ -\frac{1}{4}x_{kh}, & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} \sum_{i=0}^{k-1} \left(x_{(i+1)h} - \frac{1}{2}x_{ih} + u_{ih} \right)^2 &> \sum_{i=0}^{k-1} (x_{(i+1)h} - x_{ih} - u_{ih})^2 \\ \Rightarrow u_{kh} &= -\frac{1}{4}x_{kh} \\ \Rightarrow \left(x_{(k+1)h} - \frac{1}{2}x_{kh} + u_{kh} \right)^2 &= (x_{(k+1)h} - x_{kh} - u_{kh})^2 \\ \Rightarrow \sum_{i=0}^k \left(x_{(i+1)h} - \frac{1}{2}x_{ih} + u_{ih} \right)^2 &> \sum_{i=0}^k (x_{(i+1)h} - x_{ih} - u_{ih})^2 \\ \Rightarrow u_{(k+1)h} &= -\frac{1}{4}x_{(k+1)h} \\ \Rightarrow \dots & \\ \Rightarrow u_{Kh} &= -\frac{1}{4}x_{Kh}, \quad \text{for all } K \geq k. \end{aligned} \quad (22)$$

We can see that if at some time instant $t=kh$, the parameter estimate is $\hat{\theta}_{kh} = 2$, then it will remain $\hat{\theta}_{kh} = 2$ thereafter, and the adaptive control law will keep being $u_{Kh} = -\frac{1}{4}x_{Kh}$ for all $K \geq k$.

In fact, this may happen in a comparatively large probability which partly depends on the sample size h . Suppose that the true parameter is $(-2 \ln 2, -4 \ln 2)$ and the initial values are $x_0=1$ and $u_0=0$. Then, we have

$$\begin{aligned} u_h = -\frac{1}{4}x_h &\Leftrightarrow \left(x_h - \frac{1}{2}x_0 + u_0 \right)^2 > (x_h - x_0 - u_0)^2 \\ &\Leftrightarrow \left(\int_0^{0.5} 2^{2s-1} dW_s \right)^2 > \left(\int_0^{0.5} 2^{2s-1} dW_s - \frac{1}{2} \right)^2 \\ &\Leftrightarrow \left(\int_0^{0.5} 2^{2s} dW_s \right)^2 > \left(\int_0^{0.5} 2^{2s} dW_s - 1 \right)^2 \\ &\Leftrightarrow \int_0^{0.5} 2^{2s} dW_s > \frac{1}{2}. \end{aligned} \quad (23)$$

Since $W = \int_0^{0.5} 2^{2s} dW_s$ obeys the norm distribution $N(0, 3/4 \ln 2)$, $W > \frac{1}{2}$ occurs with probability 0.3154. Thus, the adaptive control law will stick at the “worse” control law $u_{kh} = -\frac{1}{4}x_{kh}$ at least with probability 0.3154. In this case, the cost of the corresponding discrete-time system is going far away from the optimal value if the true value is $\theta_0 = 1$. Precisely, by (16), the

optimal cost of the corresponding discrete-time system is 0.3494. However, by MatLab program, the cost under adaptive control law $\{u_{kh} = -\frac{1}{4}x_{kh}, k = 1, 2, \dots\}$ mostly falls into the interval (0.49, 0.51), which is apparently larger than the optimal cost. Here due to the existence of Brownian motion and the limitation of sample step, the cost under adaptive control law is random. Thinking roughly, the cost of the SD-based system will also go far away from the optimal value.

Actually, the least square algorithm has a natural tendency to return estimates with larger optimal cost. In a general way, suppose $\hat{\theta}_{kh}$ is an estimate of θ_0 at $t=kh$ given by the least square method, and $u_{kh} = \bar{K}(\hat{\theta}_{kh})x_{kh}$ is the adaptive control where $\bar{K}(\hat{\theta}_{kh})$ is defined by (12) with $\theta = \hat{\theta}_{kh}$. Then, just as Campi and Kumar (1998) pointed out, what the least square algorithm returns is the most possible closed-loop system, and the behaviour of the true system $(\bar{A}(\theta_0), \bar{B}(\theta_0))$ with the loop closed by adaptive control law $u_{kh} = \bar{K}(\hat{\theta}_{kh})x_{kh}$ is the same closed-loop system with estimate $(\bar{A}(\hat{\theta}_{kh}), \bar{B}(\hat{\theta}_{kh}))$, in other words, their closed-loop gains are the same and equal to

$$\bar{A}(\theta_0) + \bar{B}(\theta_0)\bar{K}(\hat{\theta}_{kh}) = \bar{A}(\hat{\theta}_{kh}) + \bar{B}(\hat{\theta}_{kh})\bar{K}(\hat{\theta}_{kh}).$$

This means that the cost of running the true system is the same as that of the estimated system under feedback $\bar{K}(\hat{\theta}_{kh})$. Meanwhile, $\bar{K}(\hat{\theta}_{kh})x_{kh}$ is apparently not optimal for the true system, but optimal for the system $(\bar{A}(\hat{\theta}_{kh}), \bar{B}(\hat{\theta}_{kh}))$. Thus,

$$\bar{J}(\hat{\theta}_{kh}) \geq \bar{J}(\theta_0).$$

To overcome this nature tendency, similar to Kumar (1983) we will adopt a cost-biased method to estimate the unknown parameter and design SD-based adaptive control.

3.1 Parameter estimation and SD-based adaptive control design

This subsection is devoted to designing an SD-based adaptive control. We first use a cost-biased estimator to estimate the unknown parameter θ_0 , and then, use the certainty equivalent principle to construct the desired adaptive control.

Similar to Kumar (1983), choose an arbitrary deterministic function $\varphi(k)$ such that

$$\varphi(k) > 0, \quad \lim_{k \rightarrow \infty} \varphi(k) = +\infty, \quad \lim_{k \rightarrow \infty} \frac{\varphi(k)}{\ln k} = 0. \quad (24)$$

At each time $t = kh$, estimate the unknown parameter θ_0 as follows:

$$\hat{\theta}_{kh} = \begin{cases} \underset{\theta \in \Theta}{\operatorname{argmin}}[\varphi(k)\bar{J}(\theta) + V_k(\theta)], & k \text{ is even,} \\ \hat{\theta}_{(k-1)h}, & k \text{ is odd,} \end{cases} \quad (25)$$

where $\bar{J}(\theta)$ is given by (16), and

$$V_k(\theta) = \sum_{i=0}^{k-1} (x_{(i+1)h} - \bar{A}(\theta)x_{ih} - \bar{B}(\theta)u_{ih})^T \times (x_{(i+1)h} - \bar{A}(\theta)x_{ih} - \bar{B}(\theta)u_{ih}). \quad (26)$$

Define the SD-based adaptive control as

$$u_t = u_{t'}, \quad t \in [t', t' + h), \quad t' = \lfloor \frac{t}{h} \rfloor h; \quad (27)$$

$$u_{kh} = \bar{K}(\hat{\theta}_{kh})x_{kh}, \quad (28)$$

where $\bar{K}(\hat{\theta}_{kh})$ is given by (12) and (25), or

$$\bar{K}(\hat{\theta}_{kh}) = -[\bar{B}^T(\hat{\theta}_{kh})\bar{P}(\hat{\theta}_{kh})\bar{B}(\hat{\theta}_{kh}) + R]^{-1} \times \bar{B}^T(\hat{\theta}_{kh})\bar{P}(\hat{\theta}_{kh})\bar{A}(\hat{\theta}_{kh}). \quad (29)$$

3.2 Convergence analysis of parameter estimates

Let

$$D_k(\theta) = \varphi_k \bar{J}(\theta) + V_k(\theta), \quad (30)$$

$$\phi_{ih}(\theta) = [\bar{A}_0 - \bar{A}(\theta)]x_{ih} + [\bar{B}_0 - \bar{B}(\theta)]u_{ih}, \quad (31)$$

$$\mu_0(\theta) = 1, \quad \mu_k(\theta) = 1 + \sum_{i=1}^{k-1} \phi_{ih}^T(\theta)\phi_{ih}(\theta), \quad (32)$$

where $\bar{A}_0 = \bar{A}(\theta_0)$, $\bar{B}_0 = \bar{B}(\theta_0)$, and $\bar{J}(\theta)$ is given by (15) and (16). Then, by (15) we have

$$V_k(\theta) = \mu_k(\theta) + 2 \sum_{i=0}^{k-1} \phi_{ih}^T(\theta)\bar{W}_{(i+1)h}(\theta_0) + \sum_{i=0}^{k-1} \bar{W}_{(i+1)h}^T(\theta_0)\bar{W}_{(i+1)h}(\theta_0). \quad (33)$$

Before going further, we need the following lemmas.

Lemma 1: *If θ^* is a limit point of $\{\hat{\theta}_{kh}\}_{k=1}^\infty$ almost surely, and the sample step size h is such that $n\sigma_h^2 \leq 1$, then $\bar{J}(\theta^*) \leq \bar{J}(\theta_0)$, where $\bar{J}(\theta)$ is defined by (15), and*

$$\sigma_h = \left\| \int_0^h e^{A_0 s} C_0 (e^{A_0 s} C_0)^T ds \right\|^{1/2}. \quad (34)$$

The proof is given in Appendix A.

Remark 1: To ensure $n\sigma_h^2 \leq 1$, by (34) it suffices to choose h such that

$$h \leq \min \left\{ 1, \frac{1}{\sqrt{n} \|C_0\|^2 e^{2\|A_0\|}} \right\}.$$

In fact, we know that, under Assumption A3 and the condition $n\sigma_h^2 \leq 1$, h should belong to set

$$\mathcal{H} = \left\{ h : h < \min_{\substack{\lambda \in \Lambda(A(\theta)) \\ \theta \in \Theta}} \left\{ 1, \frac{1}{\sqrt{n} \|C_0\|^2 e^{2\|A_0\|}}, \frac{\pi}{|\operatorname{Im}\lambda|} \right\} \right\}. \quad (35)$$

Lemma 2: Under Assumptions A1–A3 and the condition $n\sigma_h^2 \leq 1$, if for a $\theta^* \in \Theta$, the parameter estimate $\hat{\theta}_{kh}$ given by (25) satisfies

$$\limsup_{k \rightarrow \infty} \frac{1}{\ln k} \sum_{i=0}^{k-1} 1(\hat{\theta}_{ih} = \theta^*) > 0 \quad a.s., \quad (36)$$

then

$$\bar{A}_0 + \bar{B}_0 \bar{K}(\theta^*) = \bar{A}(\theta^*) + \bar{B}(\theta^*) \bar{K}(\theta^*), \quad (37)$$

where $\bar{A}_0 = \bar{A}(\theta_0)$, $\bar{B}_0 = \bar{B}(\theta_0)$, and $\bar{A}(\theta)$, $\bar{B}(\theta)$ and $\bar{K}(\theta)$ are given by (8) and (12), respectively.

The proof is given in Appendix A.

From Lemmas 1 and 2, and similar to the proof of Theorem 8 in Kumar (1983), we can get the following theorem.

Theorem 1: Suppose that Assumptions A1–A3 hold and the sample step size h is such that $n\sigma_h^2 \leq 1$, where σ_h is defined by (34). Then, under the SD-based adaptive control (24)–(28), the discretized closed-loop system

$$x_{(k+1)h} = (\bar{A}_0 + \bar{B}_0 \bar{K}(\hat{\theta}_{kh}))x_{kh} + \bar{W}_{(k+1)h}(\theta_0)$$

of the linear continuous-time system (1) has the following properties

$$\lim_{k \rightarrow \infty} (\ln k)^{-1} \sum_{i=0}^{k-1} 1(\bar{K}(\hat{\theta}_{ih}) \neq \bar{K}_0) = 0 \quad a.s., \quad (38)$$

$$\lim_{k \rightarrow \infty} (\ln k)^{-1} \sum_{i=0}^{k-1} 1(u_{ih} \neq \bar{K}_0 x_{ih}) = 0 \quad a.s., \quad (39)$$

where $\bar{K}_0 = \bar{K}(\theta_0)$ is the optimal feedback gain defined by (11) and (12) to minimize the quadratic cost function (14).

Remark 2: This theorem says that the feedback gain $\bar{K}(\hat{\theta}_{kh})$ given by (24)–(28) converges in the sense of (38) and (39) to the optimal feedback gain $\bar{K}(\theta_0)$ of the discretized system

$$x_{(k+1)h} = \bar{A}_0 x_{kh} + \bar{B}_0 u_{kh} + \bar{W}_{(k+1)h}(\theta_0) \quad (40)$$

with quadratic cost function (14).

4. Stability result

The purpose of this section is to analyse the stability of the closed-loop system of the system (1) with the SD-based adaptive control (24)–(28).

Theorem 2: Consider the system (1) with normal distributed initial value x_0 satisfying $E\|x_0\|^2 < \infty$. Then, under the conditions of Theorem 1 and the SD-based adaptive control (24)–(28), we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|x_s\|^2 ds < \infty \quad a.s.. \quad (41)$$

Proof: Substituting the SD-based adaptive control (24)–(28) into (1), we get the following closed-loop system:

$$\begin{aligned} dx_t &= A_0 x_t dt + B_0 \bar{K}(\hat{\theta}_{t'}) x_{t'} dt + C_0 dW_t \\ &= A_1 x_{t'} dt + A_0(x_t - x_{t'}) dt + C_0 dW_t, \end{aligned} \quad (42)$$

where t' is defined by (4), and

$$A_1 = A_0 + B_0 \bar{K}(\hat{\theta}_{t'}).$$

From (42) it follows that

$$\begin{aligned} x_t - x_{t'} &= A_0 \int_{t'}^t (x_s - x_{t'}) ds + (t - t') A_1 x_{t'} \\ &\quad + C_0(W_t - W_{t'}), \quad \forall t \in [t', t' + h). \end{aligned} \quad (43)$$

Hence,

$$\begin{aligned} \|x_t - x_{t'}\| &\leq \|A_0\| \int_{t'}^t \|x_s - x_{t'}\| ds + h \|A_1\| \|x_{t'}\| \\ &\quad + \|C_0(W_t - W_{t'})\|, \end{aligned}$$

and by the Gronwall lemma,

$$\begin{aligned} \|x_t - x_{t'}\| &\leq h \|A_1\| \|x_{t'}\| e^{\|A_0\|h} + \|C_0(W_t - W_{t'})\| \\ &\quad + \|A_0\| \int_{t'}^t e^{\|A_0\|(t-s)} \|C_0(W_s - W_{t'})\| ds \\ &\leq c_1 \|x_{t'}\| + c_2(t), \end{aligned} \quad (44)$$

where

$$\begin{cases} c_1 = he^{\|A_0\|h} \max_{\theta \in \Theta} \|A_0 + B_0 \bar{K}(\theta)\|, \\ c_2(t) = \|C_0(W_t - W_{t'})\| + \|A_0\|e^{\|A_0\|h} \\ \quad \times \int_{t'}^t \|C_0(W_s - W_{s'})\| ds. \end{cases}$$

Therefore, we have

$$\begin{aligned} \|x_t\| &= \|x_{t'} + x_t - x_{t'}\| \leq \|x_{t'}\| + \|x_t - x_{t'}\| \\ &\leq (c_1 + 1)\|x_{t'}\| + c_2(t). \end{aligned} \tag{45}$$

Noting that $\|W_t - W_{t'}\|$ is independent, from the law of large number it follows that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|c_2(s)\|^2 ds = E \int_0^h \|c_2(s)\|^2 ds = O(h^2). \tag{46}$$

Hence, applying the ergodic theorem to $\|\bar{W}_{kh}(\theta_0)\|^{2^n}$, by Assumptions A1–A3, (38), and Theorem 12 of Kumar (1983) we can obtain

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \|x_{ih}\|^p < \infty \quad a.s., \quad p = 2, 4, \tag{47}$$

which together with (45) and (46) renders

$$\begin{aligned} &\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|x_s\|^2 ds \\ &= \limsup_{k \rightarrow \infty} \frac{1}{kh} \sum_{i=0}^{k-1} \int_{ih}^{(i+1)h} \|x_s\|^2 ds \\ &\leq \limsup_{k \rightarrow \infty} \frac{1}{kh} \sum_{i=0}^{k-1} \int_{ih}^{(i+1)h} [2(c_1 + 1)^2 \|x_{ih}\|^2 + 2c_2^2(s)] ds \\ &= \limsup_{k \rightarrow \infty} 2(c_1 + 1)^2 \frac{1}{k} \sum_{i=0}^{k-1} \|x_{ih}\|^2 + O(h^2) < \infty. \end{aligned}$$

Thus, (41) is true.

5. Optimality results

In this section, we would like to study the optimality of the SD-based adaptive control. To do so, we need the following lemma.

Lemma 3: *Let $P(\theta)$ be the positive definite solution of the continuous Riccati equation (10), and $\bar{P}(\theta)$ be the positive definite solution of the discrete Riccati equation (11) with $\bar{A}(\theta)$ and $\bar{B}(\theta)$ given by (8). Then, under Assumptions A2 and A3, we have*

$$\lim_{h \rightarrow 0} h\bar{P}(\theta) = P(\theta), \quad \lim_{h \rightarrow 0} \bar{K}(\theta) = K(\theta), \tag{48}$$

where the feedback gains $K(\theta)$ and $\bar{K}(\theta)$ are defined by (12).

Proof: From (11) it follows that

$$\begin{aligned} \bar{P}(\theta) &= (\bar{A}(\theta) - I + I)^T [\bar{P}(\theta) - \bar{P}(\theta)\bar{B}(\theta)(\bar{B}^T(\theta)\bar{P}(\theta)\bar{B}(\theta) \\ &\quad + R)^{-1}\bar{B}^T(\theta)\bar{P}(\theta)] (\bar{A}(\theta) - I + I) + Q \\ &= (\bar{A}(\theta) - I)^T \bar{P}(\theta) (\bar{A}(\theta) - I) + (\bar{A}(\theta) - I)^T \bar{P}(\theta) \\ &\quad + \bar{P}(\theta) (\bar{A}(\theta) - I) + \bar{P}(\theta) \\ &\quad - \bar{A}^T(\theta)\bar{P}(\theta)\bar{B}(\theta)(\bar{B}^T(\theta)\bar{P}(\theta)\bar{B}(\theta) + R)^{-1} \\ &\quad \times \bar{B}^T(\theta)\bar{P}(\theta)\bar{A}(\theta) + Q, \end{aligned}$$

or equivalently,

$$\begin{aligned} 0 &= \frac{(\bar{A}(\theta) - I)^T}{h} (h\bar{P}(\theta)) + h\bar{P}(\theta) \frac{(\bar{A}(\theta) - I)}{h} + Q \\ &\quad + \frac{(\bar{A}(\theta) - I)^T}{h} (h\bar{P}(\theta)) (\bar{A}(\theta) - I) \\ &\quad - \bar{A}^T(\theta)(h\bar{P}(\theta)) \frac{\bar{B}(\theta)}{h} \left(\frac{\bar{B}^T(\theta)}{h} (h\bar{P}(\theta))\bar{B}(\theta) + R \right)^{-1} \\ &\quad \times \frac{\bar{B}^T(\theta)}{h} (h\bar{P}(\theta))\bar{A}(\theta). \end{aligned} \tag{49}$$

Denote $P_h(\theta) = h\bar{P}(\theta)$, then, (49) can be rewritten as

$$\begin{aligned} 0 &= \frac{(\bar{A}(\theta) - I)^T}{h} P_h(\theta) + P_h(\theta) \frac{(\bar{A}(\theta) - I)}{h} + Q \\ &\quad + \frac{(\bar{A}(\theta) - I)^T}{h} P_h(\theta) (\bar{A}(\theta) - I) \\ &\quad - \bar{A}^T(\theta) P_h(\theta) \frac{\bar{B}(\theta)}{h} \left(\frac{\bar{B}^T(\theta)}{h} P_h(\theta)\bar{B}(\theta) + R \right)^{-1} \\ &\quad \times \frac{\bar{B}^T(\theta)}{h} P_h(\theta)\bar{A}(\theta). \end{aligned}$$

By (8) we know that $\bar{A}(\theta)$, $\bar{B}(\theta)$, $(\bar{A}(\theta) - I)/h$ and $(\bar{B}(\theta))/h$ are continuous with respect to $h \in [0, 1]$, and

$$\left\{ \begin{aligned} \lim_{h \rightarrow 0} \bar{A}(\theta) &= \lim_{h \rightarrow 0} e^{A(\theta)h} = I, \\ \lim_{h \rightarrow 0} \bar{B}(\theta) &= \lim_{h \rightarrow 0} \int_0^h e^{A(\theta)t} dt B(\theta) = 0, \\ \lim_{h \rightarrow 0} \frac{\bar{A}(\theta) - I}{h} &= \lim_{h \rightarrow 0} \left(A(\theta) + \sum_{n=2}^{\infty} \frac{A(\theta)^n h^{n-1}}{n!} \right) = A(\theta), \\ \lim_{h \rightarrow 0} \frac{\bar{B}(\theta)}{h} &= \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h e^{A(\theta)t} dt B(\theta) \\ &= \lim_{h \rightarrow 0} \left(B(\theta) + \sum_{n=1}^{\infty} \frac{A^n(\theta) h^n}{(n+1)!} B(\theta) \right) = B(\theta). \end{aligned} \right. \tag{50}$$

By Assumption A2, the continuous Riccati equation (10) has a unique positive definite solution $P(\theta)$. Notice that the solution of the algebra Riccati equation is continuously dependent on the coefficient matrices. Then, it follows from (50) that

$$\lim_{h \rightarrow 0} P_h(\theta) = P(\theta),$$

which together with (12) (the definitions of $K(\theta)$ and $\bar{K}(\theta)$) gives

$$\begin{aligned} \bar{K}(\theta) &= -\left[\frac{\bar{B}^T(\theta)}{h}P_h(\theta)\bar{B}(\theta) + R\right]^{-1}\frac{B^T(\theta)}{h}P_h(\theta)\bar{A}(\theta) \\ &= -R^{-1}B(\theta)P(\theta)(I + o(1)). \end{aligned}$$

Thus, (48) is true. \square

Remark 3: Lemma 3 shows that when the sample step size h is small, the positive definite solution $\bar{P}(\theta)$ of discrete Riccati equation (11) is approximately equal to $(1/h)P(\theta)$. This means the smaller the h is, the larger the $\bar{P}(\theta)$ is. On the contrary, when h is small, so is the difference between the two optimal feedback gains of the discretized system (40) and the continuous-time system (1). And by Theorem 1, as $h \rightarrow 0$, the feedback gain $\bar{K}(\hat{\theta}_{kh})$ of the SD-based adaptive control (24)–(28) approaches to the optimal feedback gain K_0 of the system (1) in the sense of (38) and (39).

We now give a simple example to illustrate (48).

Example 1: Consider the system (1) with $n = m = l = 1$ and cost function (3). Let $A = 1, B = 1, Q = 1$ and $R = 1$. Then, the continuous Riccati equation (10) becomes

$$-P^2 + 2P + 1 = 0,$$

which has a unique positive solution

$$P = \sqrt{2} + 1. \tag{51}$$

For a given sample step size h , the parameters of the corresponding discretized system (11) are $\bar{A} = e^h, \bar{B} = e^h - 1$, respectively. Hence, the discrete Riccati equation (11) is

$$\bar{P} = e^{2h}\left[\bar{P} - \bar{P}^2(e^h - 1)^2((e^h - 1)^2\bar{P} + 1)^{-1}\right] + 1,$$

which has a unique positive solution

$$\bar{P} = \frac{e^h + \sqrt{e^{2h} + 1}}{e^h - 1}. \tag{52}$$

This together with (12) gives

$$K = -(\sqrt{2} + 1), \quad \bar{K} = -\frac{e^{2h} + e^h\sqrt{e^{2h} + 1}}{(e^h - 1)(e^h + \sqrt{e^{2h} + 1}) + 1}.$$

Thus, when h is sufficiently small, we have

$$\begin{aligned} h\bar{P} &= \sqrt{2} + 1 + O(h) = P + O(h), \\ \bar{K} &= -(\sqrt{2} + 1) + O(h) = K + O(h), \end{aligned}$$

i.e., (48) holds.

With Lemma 3, similar to Kumar (1983), we can prove Theorem 3.

Theorem 3: Under the condition of Theorem 2 and the SD-based adaptive control (24)–(28), for the discretized system

$$x_{(k+1)h} = \bar{A}_0x_{kh} + \bar{B}_0u_{kh} + \bar{W}_{(k+1)h}(\theta_0) \tag{53}$$

of the system (1), we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} (x_{ih}^T Q x_{ih} + u_{ih}^T R u_{ih}) &= \bar{J}(\theta_0) \\ &= \text{tr}(C_0^T P_0 C_0) + o(1) \quad a.s., \end{aligned} \tag{54}$$

where $\bar{J}(\theta_0)$ is given by (15) and (16) with $\theta = \theta_0$, i.e.,

$$\bar{J}(\theta_0) = \text{tr}\left(\int_0^h C_0^T e^{A_0^T s} \bar{P}_0 e^{A_0 s} C_0 ds\right). \tag{55}$$

Proof: From (13) we obtain

$$\begin{aligned} Q + \bar{K}^T(\hat{\theta}_{kh})R\bar{K}(\hat{\theta}_{kh}) + [\bar{A}_0 + \bar{B}_0\bar{K}(\hat{\theta}_{kh})]^T \\ \times \bar{P}_0[\bar{A}_0 + \bar{B}_0\bar{K}(\hat{\theta}_{kh})] \\ = Q + [\bar{K}_0 + (\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0)]^T R[\bar{K}_0 + (\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0)] \\ + [\bar{A}_0 + \bar{B}_0\bar{K}_0 + \bar{B}_0(\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0)]^T \bar{P}_0[\bar{A}_0 + \bar{B}_0\bar{K}_0 \\ + \bar{B}_0(\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0)] \\ = \bar{P}_0 + [\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0]^T R[\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0] + [\bar{K}(\hat{\theta}_{kh}) \\ - \bar{K}_0]^T \bar{B}_0^T \bar{P}_0 \bar{B}_0 [\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0] \\ + [\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0]^T R \bar{K}_0 + [\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0]^T \\ \times \bar{B}_0^T \bar{P}_0 [\bar{A}_0 + \bar{B}_0 \bar{K}_0] \\ + ([\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0]^T R \bar{K}_0 + [\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0]^T \\ \times \bar{B}_0^T \bar{P}_0 [\bar{A}_0 + \bar{B}_0 \bar{K}_0])^T. \end{aligned} \tag{56}$$

From the second equality of (12) we have

$$[\bar{B}_0^T \bar{P}_0 \bar{B}_0 + R] \bar{K}_0 = -\bar{B}_0^T \bar{P}_0 \bar{A}_0,$$

or equivalently,

$$R \bar{K}_0 + \bar{B}_0^T \bar{P}_0 \bar{A}_0 + \bar{B}_0^T \bar{P}_0 \bar{B}_0 \bar{K}_0 = 0,$$

which leads to

$$[\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0]^T R \bar{K}_0 + [\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0]^T \bar{B}_0^T \bar{P}_0 [\bar{A}_0 + \bar{B}_0 \bar{K}_0] = 0.$$

This together with (56) gives

$$\begin{aligned} Q + \bar{K}^T(\hat{\theta}_{kh}) R \bar{K}(\hat{\theta}_{kh}) + [\bar{A}_0 + \bar{B}_0 \bar{K}(\hat{\theta}_{kh})]^T \\ \times \bar{P}_0 [\bar{A}_0 + \bar{B}_0 \bar{K}(\hat{\theta}_{kh})] \\ = \bar{P}_0 + [\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0]^T [R + \bar{B}_0^T \bar{P}_0 \bar{B}_0] [\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0]. \end{aligned} \quad (57)$$

Let

$$\begin{aligned} y_{(k+1)} = x_{kh}^T Q x_{kh} + u_{kh}^T R u_{kh} - \bar{J}(\theta_0) + x_{(k+1)h}^T \bar{P}_0 x_{(k+1)h} \\ - x_{kh}^T \bar{P}_0 x_{kh} - x_{kh}^T [\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0]^T [R + \bar{B}_0^T \bar{P}_0 \bar{B}_0] \\ \times [\bar{K}(\hat{\theta}_{kh}) - \bar{K}_0] x_{kh}. \end{aligned} \quad (58)$$

Then, substituting

$$x_{(k+1)h} = [\bar{A}_0 + \bar{B}_0 \bar{K}(\hat{\theta}_{kh})] x_{kh} + \bar{W}_{(k+1)h}(\theta_0)$$

and $u_{kh} = \bar{K}(\hat{\theta}_{kh}) x_{kh}$

into (58) and using (57) gives

$$\begin{aligned} y_{k+1} = \bar{W}_{(k+1)h}^T(\theta_0) \bar{P}_0 \bar{W}_{(k+1)h}(\theta_0) - \bar{J}(\theta_0) + 2 \bar{W}_{(k+1)h}^T(\theta_0) \\ \times \bar{P}_0 [\bar{A}_0 + \bar{B}_0 \bar{K}(\hat{\theta}_{kh})] x_{kh}. \end{aligned} \quad (59)$$

By (55) and the definition of $\bar{W}_{kh}(\theta_0)$ we have

$$E[y_{k+1} | \mathcal{F}_{kh}] = E[\bar{W}_{(k+1)h}^T(\theta_0) \bar{P}_0 \bar{W}_{(k+1)h}(\theta_0)] - \bar{J}(\theta_0) = 0.$$

Hence, $\{y_k, \mathcal{F}_k\}$ is a martingale difference sequence.

From (59) we have

$$\begin{aligned} E[y_{k+1}^2 | \mathcal{F}_{kh}] \\ = E[\bar{W}_{(k+1)h}^T(\theta_0) \bar{P}_0 \bar{W}_{(k+1)h}(\theta_0) - \bar{J}(\theta_0)]^2 \\ + 4 x_{kh}^T [\bar{A}_0 + \bar{B}_0 \bar{K}(\hat{\theta}_{kh})]^T \bar{P}_0 E\{\bar{W}_{(k+1)h}(\theta_0) \\ \times [\bar{W}_{(k+1)h}^T(\theta_0) \bar{P}_0 \bar{W}_{(k+1)h}(\theta_0) - \bar{J}(\theta_0)]\} \\ + 4 x_{kh}^T [\bar{A}_0 + \bar{B}_0 \bar{K}(\hat{\theta}_{kh})]^T \bar{P}_0 \{E[\bar{W}_{(k+1)h}(\theta_0) \bar{W}_{(k+1)h}^T(\theta_0)]\} \\ \times \bar{P}_0 [\bar{A}_0 + \bar{B}_0 \bar{K}(\hat{\theta}_{kh})] x_{kh} \\ \leq a + c \|b\| \|x_{kh}\| + 4c^2 \sigma_h^2 \|x_{kh}\|^2 \\ \leq (a + c \|b\|) + (4c^2 \sigma_h^2 + c \|b\|) \|x_{kh}\|^2, \end{aligned} \quad (60)$$

where σ_h is given by (34), and

$$\begin{aligned} a &= E[\bar{W}_{(k+1)h}^T(\theta_0) \bar{P}_0 \bar{W}_{(k+1)h}(\theta_0) - \bar{J}(\theta_0)]^2, \\ b &= 4E\{\bar{W}_{(k+1)h}(\theta_0) [\bar{W}_{(k+1)h}^T(\theta_0) \bar{P}_0 \bar{W}_{(k+1)h}(\theta_0) - \bar{J}(\theta_0)]\}, \\ c &= \max_{\theta \in \Theta} \{\|\bar{P}_0 [\bar{A}_0 + \bar{B}_0 \bar{K}(\theta)]\|\}. \end{aligned}$$

Similar to Kumar (1983), it can be shown that under the SD-based adaptive control (24)–(28), the system state x_{kh} satisfies

$$\sum_{k=1}^{\infty} k^{-2} \|x_{kh}\|^2 < \infty \quad a.s. \quad \text{and} \quad \lim_{k \rightarrow \infty} k^{-1} \|x_{kh}\|^2 = 0 \quad a.s. \quad (61)$$

Hence, by (60) and the first inequality of (61), we have

$$\sum_{k=1}^{\infty} k^{-2} E[y_{k+1}^2 | \mathcal{F}_{kh}] < \infty \quad a.s.$$

From this, the convergence theorem of martingale difference sequence (Gong 1987), and the fact that $\{y_k, \mathcal{F}_{kh}\}$ is martingale difference sequence, it follows that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k y_i = 0 \quad a.s. \quad (62)$$

By (38) and (47) we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} x_{ih}^T [\bar{K}(\hat{\theta}_{ih}) - \bar{K}_0]^T (R + \bar{B}_0^T \bar{P}_0 \bar{B}_0) [\bar{K}(\hat{\theta}_{ih}) \\ - \bar{K}_0] x_{ih} \leq \|R + \bar{B}_0^T \bar{P}_0 \bar{B}_0\| \\ \times \limsup_{k \rightarrow \infty} \left[\frac{1}{k} \sum_{i=0}^{k-1} \|\bar{K}(\hat{\theta}_{ih}) - \bar{K}_0\|^4 \right]^{1/2} \left[\frac{1}{k} \sum_{i=0}^{k-1} \|x_{ih}\|^4 \right]^{1/2} \\ = 0 \quad a.s. \end{aligned}$$

This together with (58), (62) and the second inequality of (61) yields

$$\begin{aligned}
 0 &= \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k y_i \\
 &= \lim_{k \rightarrow \infty} \left\{ \frac{1}{k} \sum_{i=0}^{k-1} (x_{ih}^T Q x_{ih} + u_{ih}^T R u_{ih}) - \bar{J}(\theta_0) \right. \\
 &\quad + \frac{1}{k} [x_{kh}^T \bar{P}_0 x_{kh} - x_0^T \bar{P}_0 x_0] \\
 &\quad - \frac{1}{k} \sum_{i=0}^{k-1} \left[x_{ih}^T (\bar{K}(\hat{\theta}_{ih}) - \bar{K}_0)^T \right. \\
 &\quad \left. \left. \times (R + \bar{B}_0^T \bar{P}_0 \bar{B}_0) (\bar{K}(\hat{\theta}_{ih}) - \bar{K}_0) x_{ih} \right] \right\} \\
 &= \lim_{k \rightarrow \infty} k^{-1} \sum_{i=0}^{k-1} (x_{ih}^T Q x_{ih} + u_{ih}^T R u_{ih}) - \bar{J}(\theta_0),
 \end{aligned}$$

i.e.,

$$\lim_{k \rightarrow \infty} k^{-1} \sum_{i=0}^{k-1} (x_{ih}^T Q x_{ih} + u_{ih}^T R u_{ih}) = \bar{J}(\theta_0). \quad (63)$$

Notice that

$$\begin{aligned}
 &\text{tr} \left(\int_0^h C_0^T e^{A_0^T s} \bar{P}_0 e^{A_0 s} C_0 ds \right) \\
 &= \text{tr} \left(\int_0^h C_0^T (I + s A_0 + O(s^2)) (h^{-1} P_0 + o(h^{-1})) \right. \\
 &\quad \left. \times (I + s A_0 + O(s^2)) C_0 ds \right) \\
 &= \text{tr}(C_0^T P_0 C_0) + o(1),
 \end{aligned}$$

where we have used the first equality of (48) with $\theta = \theta_0$, i.e., $h\bar{P}_0 = P_0 + o(1)$.

Then, (54) follows from (63) immediately. \square

Theorem 4: Consider the system (1). Under the condition of Theorem 2 and the SD-based adaptive control (24)–(28), we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (x_s^T Q x_s + u_s^T R u_s) ds \leq \text{tr}(C_0^T P_0 C_0) + o(1). \quad (64)$$

Proof: From (24)–(28), (44) and (46) it follows that

$$\begin{aligned}
 &\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (x_s^T Q x_s + u_s^T R u_s) ds \\
 &= \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t [(x_s - x_{s'} + x_{s'})^T Q (x_s - x_{s'} + x_{s'}) + u_{s'}^T R u_{s'}] ds \\
 &= \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (x_{s'}^T Q x_{s'} + u_{s'}^T R u_{s'}) ds \\
 &\quad + \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t [2x_{s'}^T Q (x_s - x_{s'}) + (x_s - x_{s'})^T \\
 &\quad \times Q (x_s - x_{s'})] ds
 \end{aligned}$$

$$\begin{aligned}
 &\leq \lim_{k \rightarrow \infty} \frac{1}{kh} \sum_{i=0}^{k-1} \int_{ih}^{(i+1)h} (x_{ih}^T Q x_{ih} + u_{ih}^T R u_{ih}) ds \\
 &\quad + \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (2\|Q\| \|x_{s'}\| \|x_s - x_{s'}\| \\
 &\quad + \|Q\| \|x_s - x_{s'}\|^2) ds \\
 &\leq \lim_{k \rightarrow \infty} \frac{1}{kh} \sum_{i=0}^{k-1} \int_{ih}^{(i+1)h} (x_{ih}^T Q x_{ih} + u_{ih}^T R u_{ih}) ds \\
 &\quad + \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (2c_1(1 + c_1)\|Q\| \|x_{s'}\|^2 \\
 &\quad + 2\|Q\| c_2(s)\|x_{s'}\| + 2c_2^2(s)\|Q\|) ds \\
 &\leq \lim_{k \rightarrow \infty} \frac{1}{kh} \sum_{i=0}^{k-1} \int_{ih}^{(i+1)h} (x_{ih}^T Q x_{ih} + u_{ih}^T R u_{ih}) ds \\
 &\quad + \lim_{k \rightarrow \infty} \frac{1}{kh} \sum_{i=0}^{k-1} \int_{ih}^{(i+1)h} 2c_1(1 + c_1)\|Q\| \|x_{ih}\|^2 \\
 &\quad + 2\|Q\| \left(\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t c_2^2(s) ds \right)^{1/2} \\
 &\quad \times \left(\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|x_{s'}\|^2 ds \right)^{1/2} + O(h^2) \\
 &= \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} (x_{ih}^T Q x_{ih} + u_{ih}^T R u_{ih}) \\
 &\quad + 2c_1(1 + c_1)\|Q\| \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \|x_{ih}\|^2 \\
 &\quad + O(h) \left(\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \|x_{ih}\|^2 \right)^{1/2} + O(h^2). \quad (65)
 \end{aligned}$$

Notice that

$$c_1 = h e^{\|A_0\|h} \max_{\theta \in \Theta} \|A_0 + B_0 \bar{K}(\theta)\| = O(h).$$

Then, by (47), (54) and (65) we get (64). \square

Remark 5: For the system (1), the SD-based adaptive control (24)–(28) is suboptimal with respect to the quadratic index (3), since the difference between (20) and (64) is $o(1)$ when h is small.

6. Concluding remarks

By using a cost-biased least square algorithm, an SD-based adaptive LQ optimal control is designed for linear stochastic continuous-time systems with both unknown parameters and disturbances. It is worth mentioning that neither an artificial persistent nor a diminishing excitation signal is used to guarantee the optimality of the closed-loop system.

Here, what we have investigated is a comparatively simple SD-based system without time-delay or parameter variation, which should be considered in many real systems. Moreover, the noise dealt with is assumed to be a standard Brownian motion. But, in practice, bounded noises are often encountered and worth studying in detail. Besides, the parameter estimation method is nonrecursive, which may be computationally unwelcome.

Acknowledgement

This work was completed when the first author pursued her PhD degree in the Academy of Mathematics and Systems Science, Chinese Academy of Sciences. The work was supported by the National Natural Science Foundation of China under grants 90405017, 60674038, and the National Key Project for Basic Research of China under grant 2002cb312205.

Appendix A: Proofs of Lemmas 1–2

Lemma A.1: For n -dimensional vector-valued random variable $W = (w_1, w_2, \dots, w_n)^T$, if all of its components w_i ($i = 1, \dots, n$) are independent of each other, and with normal distribution $N(0, \sigma_i^2)$, $i = 1, \dots, n$, then for any given positive integer j we have

$$E(W^T W)^j \leq \frac{(2j)!(n\sigma^2)^j}{2^j j!}, \tag{A.1}$$

where $\sigma = \max\{\sigma_1, \dots, \sigma_n\}$.

Proof: Note that for 1-dimensional random variable ξ with normal distribution $N(0, \sigma^2)$,

$$E\xi^n = \begin{cases} 0, & n = 2j - 1; \\ \frac{(2j)!\sigma^{2j}}{2^j j!}, & n = 2j. \end{cases} \tag{A.2}$$

Then we have

$$\begin{aligned} E(W^T W)^j &= E(w_1^2 + \dots + w_n^2)^j \\ &= \sum_{j_1 + \dots + j_n = j} \frac{j!}{j_1! \dots j_n!} E(w_1^{2j_1} \dots w_n^{2j_n}) \\ &= \sum_{j_1 + \dots + j_n = j} \frac{j!}{j_1! \dots j_n!} Ew_1^{2j_1} \dots Ew_n^{2j_n} \\ &= \sum_{j_1 + \dots + j_n = j} \frac{j!}{j_1! \dots j_n!} \frac{(2j_1)!\sigma_1^{2j_1}}{2^{j_1} j_1!} \dots \frac{(2j_n)!\sigma_n^{2j_n}}{2^{j_n} j_n!} \end{aligned}$$

$$\begin{aligned} &= \frac{(2j)!}{2^j j!} \sum_{j_1 + \dots + j_n = j} \frac{2^{j_1} j_1! (2j_1 - 1)!!}{(j_1!)^2} \dots \frac{2^{j_n} j_n! (2j_n - 1)!!}{(j_n!)^2} \\ &\quad \times \frac{(j!)^2}{2^j j! (2j - 1)!!} (\sigma_1)^{2j_1} \dots (\sigma_n)^{2j_n} \\ &= \frac{(2j)!}{2^j j!} \sum_{j_1 + \dots + j_n = j} \frac{(2j_1 - 1)!!}{j_1!} \dots \frac{(2j_n - 1)!!}{j_n!} \\ &\quad \times \frac{j!}{(2j - 1)!!} (\sigma_1)^{2j_1} \dots (\sigma_n)^{2j_n}. \end{aligned}$$

This together with

$$\begin{aligned} &\frac{(2j_1 - 1)!!}{j_1!} \dots \frac{(2j_n - 1)!!}{j_n!} \frac{j!}{(2j - 1)!!} \\ &= \frac{j_1! (2j_1 - 1)!!}{j_1! (2j_1 - 1)!!} \dots \frac{(j_2 + j_1) \dots (j_1 + 1) (2j_2 - 1)!!}{j_2! (2j_1 + 2j_2 - 1) \dots (2j_1 + 1)} \dots \\ &\quad \times \frac{j \dots (j - j_n + 1) (2j_n - 1)!!}{j_n! (2j - 1) \dots (2j - 2j_n + 1)} \leq 1 \end{aligned}$$

results in

$$\begin{aligned} &\sum_{j_1 + \dots + j_n = j} \frac{j!}{j_1! \dots j_n!} Ew_1^{2j_1} \dots Ew_n^{2j_n} \\ &\leq \frac{(2j)!}{2^j j!} \sum_{j_1 + \dots + j_n = j} \sigma^{2j} = \frac{(2j)! (j + n - 1)!}{2^j j! (n - 1)! j!} \sigma^{2j}. \tag{A.3} \end{aligned}$$

Noticing that $(a + 1)/(b + 1) < a/b$ for all $a > b > 0$, we have

$$\begin{aligned} \frac{(j + n - 1)!}{(n - 1)! j!} &= \frac{j + n - 1}{j} \frac{j + n - 2}{j - 1} \dots \frac{n}{1} \leq n^j, \\ \forall j &= 1, 2, \dots \end{aligned}$$

□

Substituting this into (A.3) gives the desired result (A.1).

Proof of Lemma 1: By (9) we have $\overline{W}_{(k+1)h}(\theta_0) = \int_{kh}^{(k+1)h} e^{A_0(kh+h-s)} C_0 ds$, and hence,

$$E[\overline{W}_{(k+1)h}(\theta_0) \overline{W}_{(k+1)h}^T(\theta_0)] = \int_0^h e^{A_0 s} C_0 (e^{A_0 s} C_0)^T ds.$$

So, there exists a deterministic orthogonal matrix H_h such that

$$\begin{aligned} &E[H_h \overline{W}_{(k+1)h}(\theta_0) \overline{W}_{(k+1)h}^T(\theta_0) H_h^T] \\ &= \begin{pmatrix} \sigma_{1h}^2 & & \\ & \ddots & \\ & & \sigma_{nh}^2 \end{pmatrix} \leq \sigma_h^2 I. \tag{A.4} \end{aligned}$$

From the proof of Lemma 12.3 of Chen and Guo (1991), we know that for an adapted process $\{a_s, \mathcal{F}_t\}$ with $\int_0^t |a_s|^2 ds < \infty$ a.s., $\forall t \geq 0$, if $\{w_t, \mathcal{F}_t\}$ is a Brownian motion, then there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and a Brownian motion $\{\tilde{w}_t, \tilde{\mathcal{F}}_t\}$ on it such that

$$\int_0^t a_s dw_s = \tilde{w}_{b(t)},$$

where $b(t) = \int_0^t |a_s|^2 ds$. Thus, when a_s is deterministic, then $\int_0^t a_s dw_s$ is with normal distribution. Let $H_h \bar{W}_{(k+1)h}(\theta_0) = (\bar{w}_1, \dots, \bar{w}_n)^T$. Then, from (9) it follows that \bar{w}_i ($i = 1, \dots, n$) is a linear combination of independent random variables with normal distribution, and so, is with normal distribution. Furthermore, by (A.4) we see that \bar{w}_i is independent of \bar{w}_j whenever $i \neq j$. Hence, for any given positive integer j , by Lemma A.1 and the condition that $n\sigma_h^2 \leq 1$ we have

$$\begin{aligned} & E(\bar{W}_{(k+1)h}^T(\theta_0) \bar{W}_{(k+1)h}(\theta_0))^j \\ &= E(\bar{W}_{(k+1)h}^T(\theta_0) H_h^T H_h \bar{W}_{(k+1)h}(\theta_0))^j \leq \frac{(2j)!(n\sigma_h^2)^j}{2^j j!} \leq \frac{(2j)!}{2^j j!}. \end{aligned}$$

This together with (A.2) gives

$$\begin{aligned} & E\left[\sum_{j=0}^{\infty} \frac{(-\bar{W}_{(k+1)h}^T(\theta_0) \phi_{kh}(\theta))^j}{j!} \middle| \mathcal{F}_{kh}\right] \\ &= E\left[\sum_{j=0}^{\infty} \frac{(-\bar{W}_{(k+1)h}^T(\theta_0) H_h^T H_h \phi_{kh}(\theta))^j}{j!} \middle| \mathcal{F}_{kh}\right] \\ &= E\left[\sum_{j=0}^{\infty} \frac{(\bar{W}_{(k+1)h}^T(\theta_0) H_h^T H_h \phi_{kh}(\theta))^{2j}}{(2j)!} \middle| \mathcal{F}_{kh}\right] \\ &\leq E\left[\sum_{j=0}^{\infty} \frac{\|\bar{W}_{(k+1)h}(\theta_0)\|^{2j} \|\phi_{kh}(\theta)\|^{2j}}{(2j)!} \middle| \mathcal{F}_{kh}\right] \\ &= \sum_{j=0}^{\infty} \left[\|\phi_{kh}(\theta)\|^{2j} E \frac{\|\bar{W}_{(k+1)h}(\theta_0)\|^{2j}}{(2j)!} \right] \\ &\leq \sum_{j=0}^{\infty} \left[\|\phi_{kh}(\theta)\|^{2j} \frac{(2j)!}{2^j j! (2j)!} \right] \\ &= e^{1/2 \phi_{kh}^T(\theta) \phi_{kh}(\theta)}, \end{aligned} \tag{A.5}$$

where and whereafter

$$\mathcal{F}_{kh} = \sigma\{x_0, W_t, t \in [0, kh]\}. \tag{A.6}$$

It is obvious that x_{kh} , u_{kh} , $\bar{W}_{kh}(\theta_0)$, $\hat{\theta}_{kh}$, and $\phi_{kh}(\theta)$ (for any given θ) are \mathcal{F}_{kh} -measurable.

Thus, by (A.5) we have

$$\begin{aligned} & E\left[\exp\left\{-\frac{1}{2} \phi_{kh}^T(\theta) \phi_{kh}(\theta) - \bar{W}_{(k+1)h}^T(\theta_0) \phi_{kh}(\theta)\right\} \middle| \mathcal{F}_{kh}\right] \\ &= e^{-1/2 \phi_{kh}^T(\theta) \phi_{kh}(\theta)} E\left[e^{-\bar{W}_{(k+1)h}^T(\theta_0) \phi_{kh}(\theta)} \middle| \mathcal{F}_{kh}\right] \\ &= e^{-1/2 \phi_{kh}^T(\theta) \phi_{kh}(\theta)} E\left[\sum_{j=0}^{\infty} \frac{(-\bar{W}_{(k+1)h}^T(\theta_0) \phi_{kh}(\theta))^j}{j!} \middle| \mathcal{F}_{kh}\right] \\ &\leq e^{-1/2 \phi_{kh}^T(\theta) \phi_{kh}(\theta)} e^{1/2 \phi_{kh}^T(\theta) \phi_{kh}(\theta)} \\ &= 1, \end{aligned}$$

which leads to

$$\begin{aligned} & E[\exp\{-\frac{1}{2}\{V_{k+1}(\theta) - V_{k+1}(\theta_0)\}\} \middle| \mathcal{F}_{kh}] \\ &= E\left[\exp\left\{-\frac{1}{2} \sum_{i=0}^k \left\{ \phi_{ih}^T(\theta) \phi_{ih}(\theta) + 2\bar{W}_{(i+1)h}^T(\theta_0) \phi_{ih}(\theta) \right\} \middle| \mathcal{F}_{kh}\right\right] \\ &= \exp\left[-\frac{1}{2}\{V_k(\theta) - V_k(\theta_0)\}\right] \\ &\quad \times E\left[\exp\left\{-\frac{1}{2} \left\{ \phi_{kh}^T(\theta) \phi_{kh}(\theta) + 2\bar{W}_{(k+1)h}^T(\theta_0) \phi_{kh}(\theta) \right\} \middle| \mathcal{F}_{kh}\right\right] \\ &\leq \exp\left[-\frac{1}{2}\{V_k(\theta) - V_k(\theta_0)\}\right]. \end{aligned}$$

This means that, for each given $\theta \in \Theta$, $\{e^{-(1/2)\{V_k(\theta) - V_k(\theta_0)\}}, \mathcal{F}_{kh}\}$ is a positive supermartingale, and thus, converges finitely almost surely, i.e.,

$$0 \leq \lim_{k \rightarrow \infty} e^{-1/2\{V_k(\theta) - V_k(\theta_0)\}} < \infty, \text{ for every } \theta \in \Theta \text{ a.s.} \tag{A.7}$$

Suppose $\bar{J}(\theta^*) > \bar{J}(\theta_0)$, then, from (24), (30) and (A.7) we would have

$$\lim_{k \rightarrow \infty} e^{-1/2\{D_k(\theta^*) - D_k(\theta_0)\}} = 0. \tag{A.8}$$

Note that $\hat{\theta}_{kh}$ as an estimate of the unknown parameter θ_0 at time instance $t = kh$ is given by $\hat{\theta}_{kh} = \arg \min_{\theta \in \Theta} D_k(\theta)$ for even k . Then there must be $D_k(\hat{\theta}_{kh}) \leq D_k(\theta_0)$ for even k . This together with the condition that Θ is finite and that θ^* is a limit of $\{\hat{\theta}_k\}$, ensures that $\hat{\theta}_{kh} = \theta^*$ for infinitely many even ks . Thus, $D_k(\theta^*) \leq D_k(\theta_0)$ infinitely often, which contradicts (A.8). Therefore, $\bar{J}(\theta^*) \leq \bar{J}(\theta_0)$ holds. \square

Proof of Lemma 2: For $\phi_{ih}(\theta)$ and $\mu_k(\theta)$ given by (31) and (32), by Theorem 2.8 of Chen and Guo (1991) we have

$$\sum_{i=0}^{k-1} \phi_{ih}^T(\theta) \bar{W}_{(i+1)h}(\theta_0) = O(\mu_k^{1/2+\varepsilon}(\theta)), \quad \forall \varepsilon > 0. \tag{A.9}$$

By (36) there is a subsequence of the even integers $\{n_k\}$ such that $\hat{\theta}_{n_k h} = \theta^*$. Similar to the proof of Lemma 3 in Kumar (1983), by (A.9) we have

$$\lim_{n \rightarrow \infty} \frac{1}{\ln n_k} \sum_{i=0}^{n_k-1} \phi_{ih}^T(\theta^*) \phi_{ih}(\theta^*) = 0 \quad a.s. \quad (\text{A.10})$$

By Assumptions A2–A3, for any given $\theta \in \Theta$, (11) has a unique positive definite solution $\bar{P}(\theta)$. Set

$$\begin{aligned} F(\theta) &= \bar{A}_0 - \bar{A}(\theta) + \bar{B}_0 \bar{K}(\theta) - \bar{B}(\theta) \bar{K}(\theta), \\ \Gamma(\theta) &= F^T(\theta) F(\theta) \end{aligned}$$

and

$$\begin{aligned} \xi_k(\theta) &= 1 + \sum_{i=1}^k \mathbf{1}(\hat{\theta}_{ih} = \theta, i \text{ is odd}), \quad \xi_0(\theta) = 1, \\ \gamma_k(\theta) &= (x_{kh}^T \Gamma(\theta) x_{kh} \wedge 1) \mathbf{1}(\hat{\theta}_{kh} = \theta, k \text{ is odd}), \end{aligned}$$

where and whereafter $x \wedge y$ denotes $\min\{x, y\}$ for any given real numbers x and y .

Note that by (25), if $\hat{\theta}_{ih} = \theta$ for i odd, then $\hat{\theta}_{(i-1)h} = \theta$. Hence, $\mathbf{1}(\hat{\theta}_{ih} = \theta, i \text{ is odd}) = \mathbf{1}(\hat{\theta}_{(i-1)h} = \theta, i \text{ is odd})$, which means $\xi_k(\theta)$ is measurable with respect to $\mathcal{F}_{(i-1)h}$. Since $(x_{ih}^T \Gamma(\theta) x_{ih} \wedge 1) - E[(x_{ih}^T \Gamma(\theta) x_{ih} \wedge 1) | \mathcal{F}_{(i-1)h}]$ is a martingale, and

$$\begin{aligned} & \sum_{i=1}^k \{\gamma_i(\theta) - E[\gamma_i(\theta) | \mathcal{F}_{(i-1)h}]\} \\ &= \sum_{i=1}^k \mathbf{1}(\hat{\theta}_{ih} = \theta, i \text{ is odd}) \\ & \quad \times ((x_{ih}^T \Gamma(\theta) x_{ih} \wedge 1) - E[(x_{ih}^T \Gamma(\theta) x_{ih} \wedge 1) | \mathcal{F}_{(i-1)h}]), \end{aligned}$$

by using Theorem 2.8 of Chen and Guo (1991) again we can see that

$$\begin{aligned} & \sum_{i=1}^k \{\gamma_i(\theta) - E[\gamma_i(\theta) | \mathcal{F}_{(i-1)h}]\} \\ &= O\left(\left(\sum_{i=0}^k \|\mathbf{1}(\hat{\theta}_{ih} = \theta, i \text{ is odd})\|^2\right)^{1/2+\varepsilon}\right) \\ &= O(\xi_k(\theta)^{1/2+\varepsilon}), \quad \forall \varepsilon > 0. \end{aligned}$$

Thereby, when $\lim_{k \rightarrow \infty} \xi_k(\theta) = \infty$, we have

$$\lim_{k \rightarrow \infty} \xi_k^{-1}(\theta) \sum_{i=1}^k \{\gamma_i(\theta) - E[\gamma_i(\theta) | \mathcal{F}_{(i-1)h}]\} = 0. \quad (\text{A.11})$$

Denote $a_{(i-1)h} = \bar{A}_0 x_{(i-1)h} + \bar{B}_0 u_{(i-1)h}$. Then, we have

$$\begin{aligned} & E[\gamma_i(\theta) | \mathcal{F}_{(i-1)h}] \\ &= E[(x_{ih}^T F^T(\theta) F(\theta) x_{ih} \wedge 1) \mathbf{1}(\hat{\theta}_i = \theta, i \text{ is odd}) | \mathcal{F}_{(i-1)h}] \\ &= \mathbf{1}(\hat{\theta}_{i-1} = \theta, i \text{ is odd}) E[(x_{ih}^T F^T(\theta) F(\theta) x_{ih} \wedge 1) | \mathcal{F}_{(i-1)h}] \\ &= \mathbf{1}(\hat{\theta}_i = \theta, i \text{ is odd}) E\left[\left((a_{(i-1)h} + \bar{W}_{ih}(\theta_0))\right)^T F^T(\theta) F(\theta) \right. \\ & \quad \left. \cdot (a_{(i-1)h} + \bar{W}_{ih}(\theta_0)) \wedge 1\right] | \mathcal{F}_{(i-1)h}] \\ &= \mathbf{1}(\hat{\theta}_i = \theta, i \text{ is odd}) \\ & \quad \times E\left[\left(a_{(i-1)h} + \bar{W}_{ih}(\theta_0)\right)^T H_h^T H_h F^T(\theta) F(\theta) \right. \\ & \quad \left. \times H_h^T H_h (a_{(i-1)h} + \bar{W}_{ih}(\theta_0)) \wedge 1\right] | \mathcal{F}_{(i-1)h}] \\ &= \mathbf{1}(\hat{\theta}_i = \theta, i \text{ is odd}) \\ & \quad \times E\left[\left(H_h a_{(i-1)h} + H_h \bar{W}_{ih}(\theta_0)\right)^T H_h F^T(\theta) F(\theta) H_h^T \right. \\ & \quad \left. \times (H_h a_{(i-1)h} + H_h \bar{W}_{ih}(\theta_0)) \wedge 1\right] | \mathcal{F}_{(i-1)h}], \quad (\text{A.12}) \end{aligned}$$

where $H_h \bar{W}_{ih}(\theta_0)$ satisfying (A.4) has been defined in the proof of Lemma 1, and is a zero mean random vector with independent normal distributed components.

By Assumption A1, we have $\int_0^h e^{A_0 s} C_0 (e^{A_0 s} C_0)^T ds > 0$, which ensures that all the σ_{ih} , $i = 1, \dots, n$, in (A.4) are positive. Similar to Lemma 5 of Kumar (1983), we can conclude that when $F(\theta) H_h^T \neq 0$, there exists $\varepsilon_1 > 0$ such that

$$\begin{aligned} & E\left[\left(H_h a_{(i-1)h} + H_h \bar{W}_{ih}(\theta_0)\right)^T H_h F^T(\theta) F(\theta) H_h^T \right. \\ & \quad \left. \times (H_h a_{(i-1)h} + H_h \bar{W}_{ih}(\theta_0)) \wedge 1\right] \geq \varepsilon_1 > 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} & E[(a_{(i-1)h} + \bar{W}_{ih}(\theta_0))^T F^T(\theta) F(\theta) (a_{(i-1)h} \\ & \quad + \bar{W}_{ih}(\theta_0)) \wedge 1] \geq \varepsilon_1 > 0. \quad (\text{A.13}) \end{aligned}$$

This together with (A.12) implies that when $F(\theta) \neq 0$,

$$E[\gamma_i(\theta) | \mathcal{F}_{(i-1)h}] \geq \varepsilon_1 \mathbf{1}(\hat{\theta}_i = \theta, i \text{ is odd}).$$

Thus, we have

$$\begin{aligned} & \xi_k^{-1}(\theta) \sum_{i=1}^k E[\gamma_i(\theta) | \mathcal{F}_{(i-1)h}] \\ & \geq \xi_k^{-1}(\theta) \sum_{i=1}^k \varepsilon_1 \mathbf{1}(\hat{\theta}_i = \theta, i \text{ is odd}) \\ & = \varepsilon_1 \xi_k^{-1}(\theta) [\xi_k(\theta) - 1], \end{aligned}$$

and hence, by (A.11) we can conclude that when $F(\theta) \neq 0$ and $\lim_{k \rightarrow \infty} \xi_k(\theta) = \infty$ a.s.,

$$\liminf_{k \rightarrow \infty} \xi_k^{-1}(\theta) \sum_{i=1}^k \gamma_i(\theta) \geq \varepsilon_1 > 0 \quad a.s. \quad (\text{A.14})$$

We are now in a position to show (37). By condition (36), there is a subsequence n_k such that

$$\lim_{k \rightarrow \infty} (\ln n_k)^{-1} \xi_{n_k}(\theta^*) > 0. \quad (\text{A.15})$$

Without loss of generality, in the sequel, we assume that $\hat{\theta}_{n_k h} = \theta^*$ for all positive integer k . Then by (28) and (31) we get $\phi_{ih}(\theta^*) = F(\theta^*)x_{ih}$ a.s. Hence,

$$\begin{aligned} & \liminf_{k \rightarrow \infty} (\ln n_k)^{-1} \sum_{i=0}^{n_k-1} \phi_{ih}^T(\theta^*) \phi_{ih}(\theta^*) \\ & \geq \liminf_{k \rightarrow \infty} (\ln n_k)^{-1} \\ & \quad \times \sum_{i=0}^{n_k-1} 1(\hat{\theta}_{ih} = \theta^*, i \text{ is odd}) \phi_{ih}^T(\theta^*) \phi_{ih}(\theta^*) \\ & = \liminf_{k \rightarrow \infty} (\ln n_k)^{-1} \\ & \quad \times \sum_{i=0}^{n_k-1} 1(\hat{\theta}_{ih} = \theta^*, i \text{ is odd}) x_{ih}^T F^T(\theta^*) F(\theta^*) x_{ih} \\ & \geq \liminf_{k \rightarrow \infty} (\ln n_k)^{-1} \sum_{i=1}^{n_k} \gamma_i(\theta^*) \\ & = \left[\lim_{k \rightarrow \infty} (\ln n_k)^{-1} \xi_{n_k}(\theta^*) \right] \liminf_{k \rightarrow \infty} \xi_{n_k}^{-1}(\theta^*) \sum_{i=1}^{n_k} \gamma_i(\theta^*). \end{aligned} \quad (\text{A.16})$$

By (A.14) and (A.15) we see that if $F(\theta^*) \neq 0$, then there would be

$$\liminf_{k \rightarrow \infty} (\ln n_k)^{-1} \sum_{i=0}^{n_k-1} \phi_{ih}^T(\theta^*) \phi_{ih}(\theta^*) > 0 \quad a.s.$$

This contradicts (A.10). Thus, $F(\theta^*) = 0$, or equivalently, (37) holds.

References

K.J. Åström and B. Wittenmark, *Computer-Controlled Systems: Theory and Design* (third edition), Beijing: Tsinghua University Press, Prentice Hall, 2002.
 B.A. Bamieh and J.B. Pearson, "A general framework for linear periodic systems with applications to H_∞ sampled-data control", *IEEE Trans. Auto. Cont.*, 37, pp. 418–435, 1992.
 D.P. Bertsekas, *Dynamic Programming and Stochastic Control*, New York: Academic Press, 1976.

P.E. Caines and H.F. Chen, "Optimal adaptive LQG control for systems with finite state process parameters", *IEEE Trans. Auto. Cont.*, 30, pp. 185–189, 1985.
 P.E. Caines and J.F. Zhang, "On the adaptive control for jump parameter systems via nonlinear filtering", *SIAM J. Cont. Optim.*, 33, pp. 1758–1777, 1995.
 M.C. Campi and P.R. Kumar, "Adaptive linear quadratic Gaussian control: the cost-biased approach revisited", *SIAM J. Cont. Optim.*, 36, pp. 1890–1907, 1998.
 H.F. Chen, "Continuous-time stochastic adaptive control stabilizing the system and minimizing the quadratic loss function", *Syst. Sci. Math. Sci.*, 8, pp. 166–181, 1995.
 H.F. Chen, T.E. Duncan and B. Pasik-Duncan, "Stochastic adaptive control for continuous time linear systems with quadratic cost", *App. Math. Optim.*, 34, pp. 113–138, 1996.
 H.F. Chen and L. Guo, *Identification and Stochastic Adaptive Control*, Boston: Birkhauser, 1991.
 T.E. Duncan, L. Guo and B. Pasik-Duncan, "Adaptive continuous-time linear quadratic Gaussian control", *IEEE Trans. Auto. Cont.*, 44, pp. 1653–1662, 1999.
 A. Fúster, "A joint criterion for reachability and observability of nonuniformly sampled discrete systems", *IEEE Trans. Auto. Cont.*, 36, pp. 1281–1284, 1991.
 A.J. Gao and B. Pasik-Duncan, "Stochastic linear quadratic adaptive control for continuous-time first-order systems", *Syst. Cont. Lett.*, 31, pp. 149–154, 1997.
 G.L. Gong, *The Introduction of Stochastic Differential Function*, Beijing: Peking University Press, 1987.
 H.Z. Hu and C.V. Hollot, "Robustness of sampled-data control systems having parametric uncertainty: a conic sector approach", *IEEE Trans. Auto. Cont.*, 38, pp. 1541–1545, 1993.
 A. Ilchmann and S. Townley, "Adaptive sampling control of high-gain stabilizable systems", *IEEE Trans. Auto. Cont.*, 44, pp. 1961–1966, 1999.
 H. Ishii and B.A. Francis, "Quadratic stabilization of sampled-data systems with quantization", *Automatica*, 39, pp. 1793–1800, 2003.
 P.R. Kumar, "Optimal Adaptive Control of Linear-quadratic-gaussian Systems", *SIAM J. Cont. Optim.*, 21, pp. 163–178, 1983.
 D. Nešić, A.R. Teel and P.V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", *Sys. Cont. Lett.*, 38, pp. 259–270, 1999.
 R. Ortega and G. Kreisselmeier, "Discrete-time model reference adaptive control for continuous-time systems using generalized sampled-data hold functions", *IEEE Trans. Auto. Cont.*, 35, pp. 334–338, 1990.
 L. Qiu and T. Chen, " H_2 -optimal design of multirate sampled-data systems", *IEEE Trans. Auto. Cont.*, 39, pp. 2506–2511, 1994.
 E.D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems* (2nd ed), New York: Springer-Verlag, 1998.
 S. Tan, J.F. Zhang and L. Yao, "Optimality analysis of adaptive sampled control of hybrid systems with quadratic index", *IEEE Trans. Auto. Cont.*, 50, pp. 1044–1051, 2005.
 H.T. Toivonen and M.F. Sâgfors, "The sampled-data H_∞ problem: a unified framework for discretization-based methods and Riccati equation solution", *Int. J. Cont.*, 66, pp. 289–309, 1997.
 L. Yao and J.F. Zhang, "Sampled-data-based LQ control of stochastic linear continuous-time systems", *Science in China (Series F)*, 33, pp. 326–339, 2003.
 C. Zhang, R.H. Middleton and R.J. Evans, "An algorithm for multirate sampling adaptive control", *IEEE Trans. Auto. Cont.*, 34, pp. 792–795, 1989.

Downloaded By: [Zhang, Ji-Feng] At: 02:23 20 September 2007